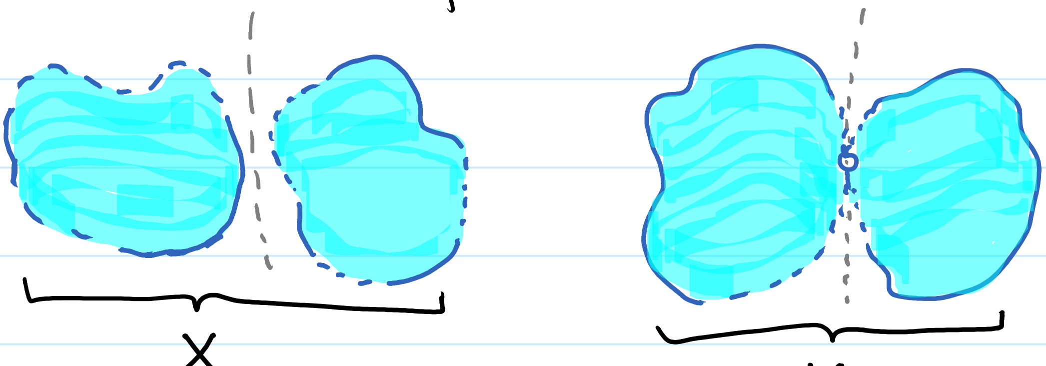
# April 6, 2017

Thursday, April 6, 2017 9:16 AM

## Natural Idea of disconnectedness




These two examples of $X \subset \mathbb{R}^2$ are disconnected cases. In both cases, one may find two "pieces", each "piece" lies in an open half of $\mathbb{R}^2$. So, each "piece" is open in $X$.

**Definition.** $(X, \mathcal{T}_X)$ is disconnected if

$$\exists\ \underbrace{\phi \neq U, V \in \mathcal{T}_X}_{\text{open wrt } X},\ \underbrace{U \cup V = X,\ U \cap V = \phi}_{\left.\begin{array}{l} V = X \setminus U \\ U = X \setminus V \end{array}\right\} \text{closed in } X}.$$

**Proposition.**

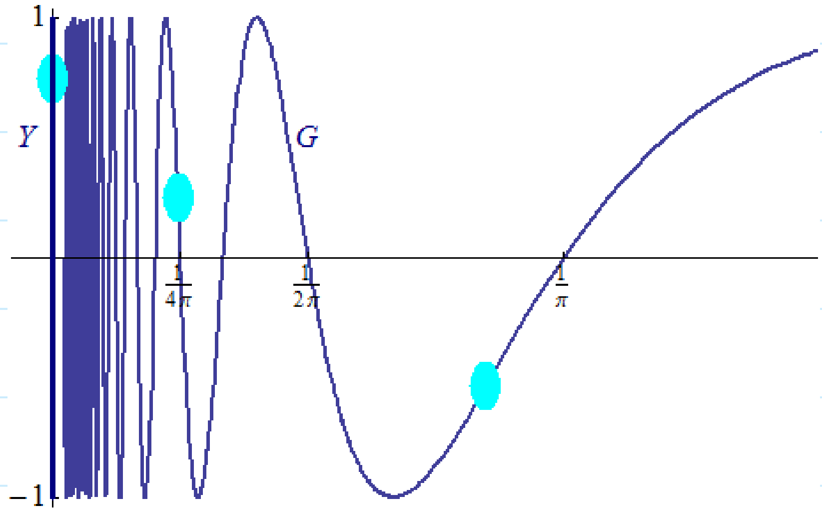
$X$ is disconnected $\Longleftrightarrow$ $\exists\ U$, both open and closed wrt $X$, $\phi \neq U \subsetneqq X$.

Concept of connectedness can be given by the negation above.

**Definition.** $(X, \mathcal{J})$ is connected if $\forall S \subset X$ which is $\underbrace{\text{both open \& closed}}_{S,\, X \setminus S \in \mathcal{J}}$ in $X$, $S = \phi$ or $S = X$

We know for any $(X, \mathcal{J})$, $\phi \in \mathcal{J}$ and $X \in \mathcal{J}$.
In other words, in a connected $X$,
$\phi$, $X = X \setminus \phi$ are the only such case.

**Famous Example.** $X = Y \cup G$ where
$Y = \{(x,y) : x = 0\}$, $G = \{(x,y) : y = \sin \frac{1}{x},\ x > 0\}$




In this $X$, $Y$ is not open as for all $U \subset \mathbb{R}^2$ with $Y \subset U \in \mathcal{J}_{std}$.)
this $U$ must contain a small nbhd of $(0,0)$ which meets $G$ infinitely many time.

From another view, $\exists$ sequence $(\frac{1}{n\pi}, 0) \in G$ but its limit $(0,0) \notin G$. $\therefore G$ is not closed.

# 04-06-p3

Thursday, April 6, 2017 9:17 AM

**Proposition.** $(X,\mathcal{J})$ is connected $\iff$

$\forall\ \phi \neq A, B \subset X$ satisfying $A \cap B = \phi$ and $A \cup B = X$,

we have $\bar{A} \cap B \neq \phi$ or $A \cap \bar{B} \neq \phi$

**Remark.** As in the example,

$\bar{Y} = Y$ and so $\bar{Y} \cap G = \phi$

But $Y \cap \bar{G} = \{0\} \times [-1,1] \neq \phi$

**Idea of Proof.**

We work on the contrapositive and assume

$\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$.

Note that $A \cap B = \phi$ and $A \cup B = X$, so $A = X \setminus B$

Moreover, $\bar{A} \cap B = \phi$ implies $\bar{A} = X \setminus B$

Thus $A = \bar{A}$, i.e., $A$ is closed

Similarly, from $A \cap \bar{B} = \phi$, get $B$ is closed

Thus, both $A, B$ are open and closed.

**Other Examples.**

* $X = (0,2)$ which is connected

$= \underbrace{(0,r]}_{A=\bar{A}} \cup \underbrace{(r,2)}_{B}$ for any $0 < r < 2$

$\bar{B} = [r,2)$ in $X$

$\therefore\ A \cap \bar{B} = \{r\} \neq \phi$

* $X = \underbrace{(0,1)}_{A=\bar{A}} \cup \underbrace{(1,2)}_{B=\bar{B}}$ in $X$

$\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$

**Definition.** Let $x_0 \in X$. $C \subset X$ is the connected component of $x_0$ if either one holds.

① $C$ is the largest connected subset of $X$ containing $x_0$.

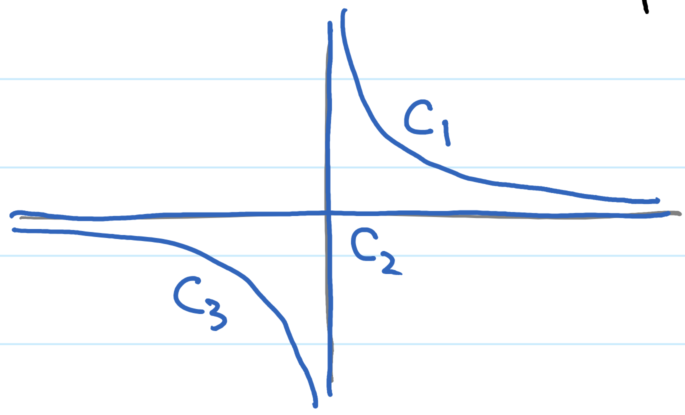
② $C = \bigcup \{A \subset X : x_0 \in A \text{ and } A \text{ is connected}\}$

③ For the equivalence relation $x \sim y$ on $X$ where $x, y \in A$ for some connected $A \subset X$; $C = [x_0]$, the equivalence class of $x_0$.

**Example.** As a subspace of standard $\mathbb{R}^2$,

$X = \{(x,y) : xy = 0 \text{ or } xy = 1\}$

It has three connected components



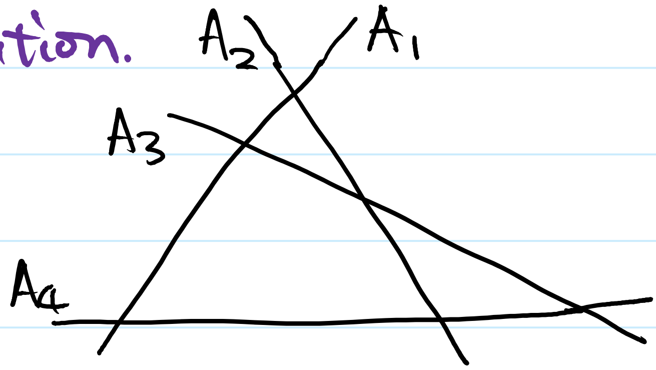

We need to make sure that the three definitions are well-defined and equivalent. The following is useful. In fact, it will be useful in many situations.

04-06-p5

Thursday, April 13, 2017 9:14 AM

**Theorem.** Let $A_\alpha \subset X$ be connected subsets. If $\forall\, \alpha, \beta \in I$, $A_\alpha \cap A_\beta \neq \phi$ then the union $A = \bigcup_{\alpha \in I} A_\alpha$ is connected

**Illustration.**




is connected

**Application** to the 3 definitions.

We have all connected subsets $A_\alpha$, $x_0 \in A_\alpha$

Then $x_0 \in \bigcap_{\alpha \in I} A_\alpha$, implies $A_\alpha \cap A_\beta \neq \phi$.

and $C = \bigcup_{\alpha \in I} A_\alpha$

**Idea of proof.** Assume $S \subset A = \bigcup_{\alpha \in I} A_\alpha$ such that $S$ is both open & closed in $A$

Then $S \cap A_\alpha$ is both open & closed in $A_\alpha$

$\therefore \forall\, \alpha \in I$, $\underbrace{S \cap A_\alpha = \phi \quad \text{or} \quad S \cap A_\alpha = A_\alpha}$

at this point, it may be $\phi$ for some $\alpha$ but $A_\beta$ for some $\beta$.

Use the condition to show

$(S \cap A_\alpha = \phi \ \forall \alpha)$ or $(S \cap A_\alpha = A_\alpha \ \forall \alpha)$

Then $S = \phi$ or $S = A$.